Method of Dimensionality Reduction and Boundary Element Method: Foundations and Applications in Contact Mechanics and Tribology

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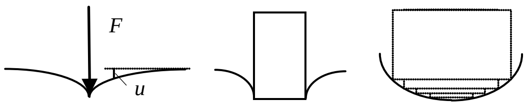
1. Introduction

We live in the age of information technologies. In structural dynamics, acoustics, electrical engineering, fluid dynamics and many other areas computer simulations became an integral part of technological development. However, numerical simulation of contact problems is still in its infancy. Simulation of "contact interface" remains in many industrial simulation programs a weak spot. The reason is the mathematical complexity of contact problems: they are described by integral equations with mixed boundary conditions, while neither the pressure nor the the deformation fields of the contacting bodies are known in advance; they have to be determined by iterations in the course of solution. Further, in many tribological problems, surface roughness may play an important role which requires very fine discretization. Standard finite element programs allow for simulation of contact problems. However, they are not fast enough to allow extensive parameter studies and they cannot be used for calculation of contact forces in superordinate system dynamics simulations.

Many mathematical methods of handling systems with interacting degrees of freedoms are based on changing the "parametrization" of the considered system in such a way that the degrees of freedom become non-interacting. One of the most effective analytical and numerical methods in Science – the Fourier analysis – is based on the replacement the parametrization with coordinates by parametrization with wave vectors which transforms partial differential equations or integral equations containing "convolutions" into algebraic ones. The same idea lies behind the "modal" reduction in structural dynamics.

The standard formulation of contact mechanical problems in the geometrically linear approximation make use of the so-called "fundamental" solution of the theory of elasticity determining the deformation of the contact place under the action of a single concentrated force (Fig.1a).



a **b** **c**

Fig.1 (a) Fundamental solution is a basis of all "standard" formulations of analytical and numerical simulation methods in contact mechanics. (b) Indentation of a flat-ended punch. (c) Jäger representation of arbitrary axis-symmetric body as superposition of flat-ended cylindrical indentations.

For the isotropic elastic half-space, this solution has the form

$$u_z = \frac{1}{\pi E^*}\frac{1}{r}F_N\,, \tag{1}$$

where u_z is the normal displacement of a surface point, F_N is the normal force, r the polar radius in the contact surface, and E^* the effective elastic modulus [1]. An arbitrary stress distribution $p(x',y')$ then leads to the surface deformation

$$u_z = \frac{1}{\pi E^*}\iint p(x',y')\frac{dx'dy'}{r}\,,\quad r=\sqrt{\left(x-x'\right)^2+\left(y-y'\right)^2}\,. \tag{2}$$

The two most powerful computational methods in contact mechanics developed in the last decade – the Boundary Element Method (BEM) and the Method of Dimensionality Reduction (MDR) are based on transformations of the integral equation (2) to an algebraic one:

- MDR uses the "Jäger superposition" [2] (Fig. 1b.c);
- BEM uses the fast Fourier-Transformation [3].

2. MDR: Fundamental solution vs Jäger superposition

If the indentation depth d as function of the contact radius *a would be known*:

$$d = g(a)\,, \tag{3}$$

then the normal force F_N as function of indentation would be trivially given by

$$F_N = \int_0^{F_N} \mathrm{d}\tilde{F}_N = \int_0^a \frac{\mathrm{d}\tilde{F}_N}{\mathrm{d}\tilde{d}}\frac{\mathrm{d}\tilde{d}}{\mathrm{d}\tilde{a}}\mathrm{d}\tilde{a} = \int_0^a \frac{\mathrm{d}k(\tilde{a})}{\mathrm{d}\tilde{a}}\left(d-g(\tilde{a})\right)\mathrm{d}\tilde{a} \tag{4}$$

where $k(\tilde{a}) = \mathrm{d}\tilde{F}_N/\mathrm{d}\tilde{d}$ is stiffness of a cylindrical punch with radius $\tilde{a}$. The beautiful feature of Eqs. (3) and (4) is that they could be interpreted as indentation of a modified profile $g(\tilde{a})$ into elastic foundation with independent springs with spacing $\mathrm{d}\tilde{a}$ having stiffness $\frac{1}{2}\frac{\mathrm{d}k(\tilde{a})}{\mathrm{d}\tilde{a}}\mathrm{d}\tilde{a}$ (Fig.2). The modified profile $g(\tilde{a})$ can be extracted from the solution for a rigid flat-ended punch (Fig.1b).

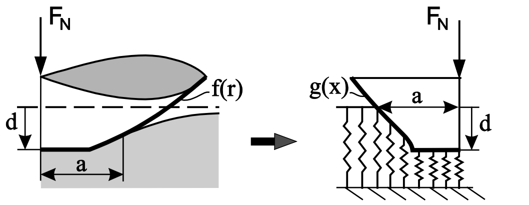


Fig. 2 MDR-transformation: The initial profile is replaced by a transformed one and at the same time the elastic half-space by the equivalent Winkler foundation.

This interpretation is the key of the Method of Dimensionality Reduction. According to Eq. (4), the application of the MDR is straightforward, if two conditions are fulfilled: (1) the contact stiffness of a cylindrical punch with radius $\tilde{a}$, $k(\tilde{a})$ is known, and there is a rule of determining the modified profile $g(\tilde{a})$. It is of no importance, in what way these two steps are done: analytically, numerically – or even experimentally. For the homogeneous media, these transformations are explicitly known. If the initial three-dimensional profile is $f(r)$ then the MDR-transformed profile is determined by the Eq.

$$g(x) = |x|\int_0^{|x|}\frac{f'(r)}{\sqrt{x^2-r^2}}\mathrm{d}r\,. \tag{5}$$

The stiffness of a contact with a cylindrical punch is $k(\tilde{a}) = 2E^*a$, thus providing the rule for the stiffness of springs of equivalent elastic foundation $\mathrm{d}k = E^*\mathrm{d}x$.

Once the solution of the equivalent one-dimensional problem is obtained, the exact three-dimensional solution is restored by a small number of simple rules [4],[5]. The solution of the MDR formally reproduces the solution first obtained by Galin [6] and later used by Sneddon in his very much cited paper [7].

Combined with reduction of Cattaneo [8] and Mindlin [9] of the tangential contact problem to the normal contact and the Lee [10] and Radok's [11] reduction of viscoelastic contact to elastic contact as well as reduction of adhesive contact to a superposition of non-adhesive solutions (JKR theory [12]) this provides a very powerful method of treating the above mentioned classes of problems. In the case of functionally graded materials, a new profile-transformation has to be used [13],[14]. The MDR can also be applied to arbitrary shaped contacts. How to do this is described in [15].

3. BEM: Coordinates vs wave vectors

The most effective method of numerical simulation of contacts of an arbitrary shape is the Boundary Element Method (BEM) [3] as it needs to discretize only the surface of contacting bodies. In the modern numerical realizations of BEM, the integral Eq. (2) in corresponding discretized form is solved in Fourier-space where it becomes simple algebraic equation. The main time consuming steps of the solution are shifted to the direct and inverse Fourier-Transformations. The two keys for acceleration these steps are:

(a) Using the Fast Fourier-Transformation (FFT), and

(b) Parallel computation on graphic cards, see for details [3].

In Fig. 3, a typical simulation of a contact of rough surface is shown which presently is possible at any desktop computer equipped with a graphic card.

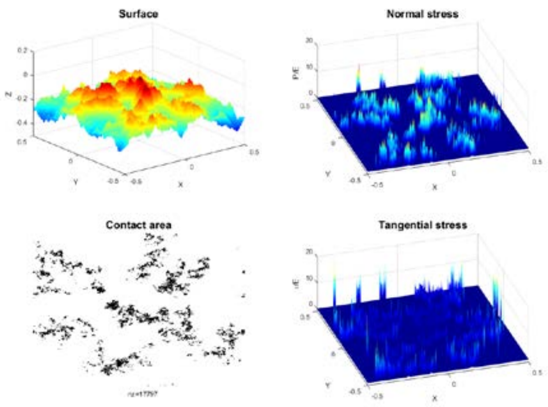


Fig.3 Rough surface, contact area, as well as normal and tangential stress distributions simulated by BEM [3].

The numerical effectiveness of BEM allows generalization it and application of BEM for analysis of normal [16] and tangential [17] contacts with viscoelastic bodies. It even allows simulation of dynamic process in systems containing viscous contacts as e.g. impacts [18].

Fig.4 Consecutive contact configuration of a flat-ended stamp in form of “Marry Poppins” calculated with adhesive BEM [20].

Simulation of adhesive contacts with BEM was for a long time a non-solved problem. The breakthrough came 2015 by formulating a mesh-dependent local detachment criterion based on the energy balance [19], which later was generalized for gradient media [20]. This formulation was validated by comparison with known exact analytical solutions and also withstands usual tests of independence of mesh size and orientation of the discretization network [20]. In [21], a large collection of simulation results for various shapes of contacting bodies is presented. Fig. 4 shows one example of this collection: Three stages of detachment of a flat punch in the form of “Marry Poppins”.

A detailed description of the adhesive BEM including validation by specially designed experiments can be found in [22].

References

[1] V.L. Popov, Contact mechanics and friction. Physical principles and applications, 2nd edition, Springer, Berlin Heidelberg, (2017).

[2] J. Jaeger, Axi-symmetric bodies of equal material in contact under torsion or shift, Archive of Applied Mechanics 65 (1995) 478-487.

[3] R. Pohrt, Q. Li, Complete boundary element formulation for normal and tangential contact problems. Physical Mesomechanics, 17 (4) (2014) 334-340.

[4] V.L. Popov, M. Heß, Method of dimensionality reduction of contact mechanics and friction, Springer, Berlin Heidelberg, (2015).

[5] V.L. Popov, M. Hess, Method of dimensionality reduction in contact mechanics and friction: a users handbook. I. Axially-symmetric contacts, Facta Univ., series Mech. Eng., 12 (1) (2014) 1-14.

[6] L.A. Galin, Contact Problems in the Theory of Elasticity, North Carolina State College, USA (1961)

[7] I.N.Sneddon, The relation between load and penetration in the axisymmetric Boussinesq problem for a punch of arbitrary profile, Int. J. Eng. Sci., 3(1) (1965) 47-57.

[8] C. Cattaneo, Sul contatto di due corpi elastici: distribuzione locale degli sforzi, Rendiconti dell'Accademia nazionale dei Lincei 27 (1938) 342-348, 434-436, 474-478.

[9] R.D. Mindlin, Compliance of elastic bodies in contact, Journal of Applied Mechanics 16 (1949) 259–268.

[10] E.H. Lee, Stress analysis in viscoelastic bodies, Quart. Appl. Math. 13 (1955) 183-190.

[11] J.R.M. Radok, Viscoelastic stress analysis, Quart. Appl. Math, 15 (1957) 198-202.

[12] K.L. Johnson, K. Kendall, A.D. Roberts, Surface Energy and the Contact of Elastic Solids. Proceedings of the Royal Society of London, Series A, 324, (1971) 301–313.

[13] M. Hess, A simple method for solving adhesive and non-adhesive axisymmetric contact problems of elastically graded materials, Int J. Eng. Science, 104 (2016) 20-35.

[14] M. Heß, V.L. Popov, Method of dimensionality reduction in contact mechanics and friction: A user’s handbook. II. Power-law graded materials. Facta Universitatis, series: Mech. Eng., 14 (3) (2016) 250-271.

[15] I. Argatov, M. Heß, R. Pohrt, and V.L. Popov, The extension of the method of dimensionality reduction to non-compact and non-axisymmetric contacts. ZAMM · Z. Angew. Math. Mech. 96, No. 10, (2016) 1144–1155.

[16] S. Kürschner, V.L. Popov, Penetration of self-affine fractal rough rigid bodies into a model elastomer having a linear viscous rheology. Physical Review E 87 (4) (2013) 042802.

[17] S. Kusche, Frictional force between a rotationally symmetric indenter and a viscoelastic half-space. ZAMM · Z. Angew. Math. Mech. 97 (2) (2017) 226–239.

[18] S. Kusche, The boundary element method for viscoelastic material applied to the oblique impact of spheres. Facta Universitatis, Series: Mechanical Engineering, 14 (3) (2016) 293-300.

[19] R. Pohrt, V.L. Popov, Adhesive contact simulation of elastic solids using local mesh-dependent detachment criterion in Boundary Elements Method. Facta Universitatis, Series: Mechanical Engineering, 13 (1) (2015) 3-10.

[20] Q. Li, V.L. Popov, Boundary element method for normal non-adhesive and adhesive contacts of power-law graded elastic materials. arXiv:1612.08395 (2016).

[21] R. Pohrt, V.L. Popov, Force-distance-dependencies for the adhesive contact of cylinders, http://dx.doi.org/10.14279/depositonce-4957 (2016).

[22] V.L. Popov, R. Pohrt, Q. Li, Strength of Adhesive Contacts: Influence of Contact Geometry and Material Gradients. Friction, (2016) to be published.